Dynamic Programming

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cs302
Spring 2013

Administative

Dynamic programming

One of the most important algorithm tools!

Very common interview question

Method for solving problems where optimal solutions can be defined in terms of optimal solutions to sub-problems

AND

the sub-problems are overlapping

Fibonacci numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, …

What is the recurrence for the n^{th} Fibonacci number?

F(n) = F(n-1) + F(n-2)

The solution for n is defined with respect to the solution to smaller problems (n-1 and n-2)

Fibonacci: a first attempt

```
FIBONACCI(n)
if n = 1 or n = 2
        return 1
else
        return FIBONACCI(n − 1) + FIBONACCI(n − 2)
```

Is it correct?

```
FIBONACCI(n)
if n = 1 or n = 2
        return 1
else
        return FIBONACCI(n − 1) + FIBONACCI(n − 2)
```

$F(n) = F(n-1) + F(n-2)$

Running time

```
FIBONACCI(n)
if n = 1 or n = 2
        return 1
else
        return FIBONACCI(n − 1) + FIBONACCI(n − 2)
```

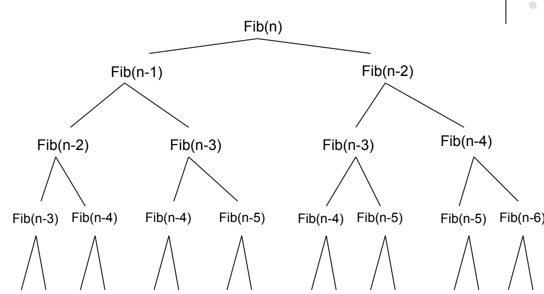
Each call creates two recursive calls

Each call reduces the size of the problem by 1 or 2

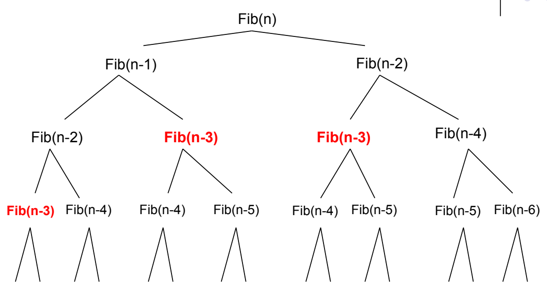
Creates a full binary of depth n

$O(2^n)$

Can we do better?



A lot of repeated work!

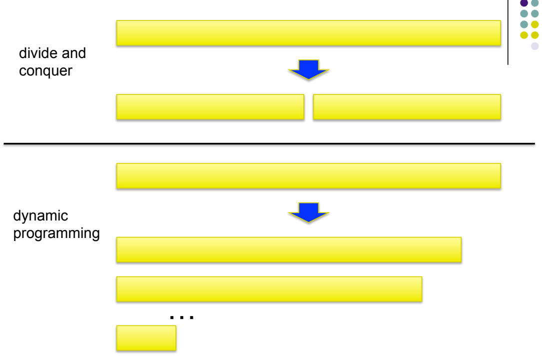


Identifying a dynamic programming problem

The solution can be defined with respect to solutions to subproblems

The subproblems created are *overlapping*, that is **we see the same subproblems repeated**

Overlapping sub-problems



Creating a dynamic programming solution

Step 1: Identify a solution to the problem with respect to **smaller** subproblems (pretend like you have a solver, but it only works on smaller problems):

- F(n) = F(n-1) + F(n-2)

Step 2: **bottom up** - start with solutions to the smallest problems and build solutions to the larger problems

use an array to store solutions to subproblems

```
FIBONACCI-DP(n)
fib[1] ← 1
fib[2] ← 1
for i ← 3 to n
        fib[i] ← fib[i − 1] + fib[i − 2]
return fib[n]
```

Is it correct?

```
FIBONACCI-DP(n)
fib[1] ← 1
fib[2] ← 1
for i ← 3 to n
        fib[i] ← fib[i − 1] + fib[i − 2]
return fib[n]
```

F(n) = F(n-1) + F(n-2)

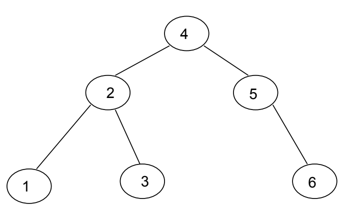
Running time?

```
FIBONACCI-DP(n)
fib[1] ← 1
fib[2] ← 1
for i ← 3 to n
        fib[i] ← fib[i − 1] + fib[i − 2]
return fib[n]
```

$\Theta(n)$

Counting binary search trees

How many unique binary search trees can be created using the numbers 1 through n?

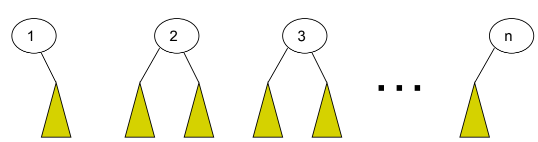


Step 1: What is the subproblem?

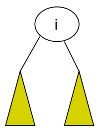
Assume we have some black box solver (call it T) that can give us the answer to smaller subproblems

How can we use the answer from this to answer our question?

How many options for the root are there?

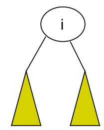


Subproblems



How many trees have i as the root?

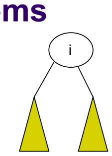
Subproblems



1, 2, …, i-1   i+1, i+2, …, n

?

Subproblems

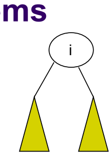


1, 2, …, i-1   i+1, i+2, …, n

T(i-1)   ?

subproblem of
size i-1

Subproblems

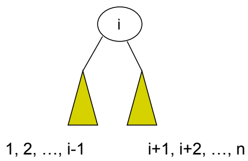


1, 2, …, i-1   i+1, i+2, …, n

T(i-1)

Number of trees for i+1, i+2, …, i+n
is the same as the number of trees
from 1, 2, …, n-i

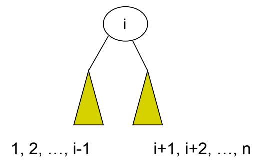
Subproblems



T(i-1)   T(n-i)

Given solutions for T(i-1) and T(n-i) how many trees are there with i as the root?

Subproblems



T(i-1)   T(n-i)

T(i) = T(i-1) * T(n-i)

Step 1: define the answer with respect to subproblems

T(i) = T(i-1) * T(n-i)

$$T(n) = \sum_{i=1}^{n} T(i-1) * T(n-i)$$

```
BST-Count(n)
  if n = 0
        return 1
  else
        sum = 0
        for i ← 1 to n
              sum ← sum + BST-Count(i − 1) ∗ BST-Count(n − i)
  return sum
```

Is there a problem?

```
BST-Count(n)
  if n = 0
        return 1
  else
        sum = 0
        for i ← 1 to n
              sum ← sum + BST-Count(i − 1) ∗ BST-Count(n − i)
  return sum
```

As with Fibonacci, we're repeating a lot of work

Step 2: Generate a solution from the bottom-up

```
BST-COUNT(n)
if n = 0
        return 1
else
        sum = 0
        for i ← 1 to n
                sum ← sum + BST-COUNT(i − 1) ∗ BST-COUNT(n − i)
return sum
```

```
BST-COUNT-DP(n)
c[0] = 1
c[1] = 1
for k ← 2 to n
        c[k] ← 0
        for i ← 1 to k
                c[k] ← c[k] + c[i − 1] ∗ c[k − i]
return c[n]
```

```
BST-COUNT-DP(n)
c[0] = 1
c[1] = 1
for k ← 2 to n
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```

0 1 2 3 4 5 … n

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0 1 2 3 4 5 … n

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for k ← 2 to n
        c[k] ← 0
        for i ← 1 to k
                c[k] ← c[k] + c[i − 1] ∗ c[k − i]
return c[n]
```

c[0]*c[1] + c[1]*c[0]

1 1

0 1 2 3 4 5 … n

```
BST-COUNT-DP(n)
c[0] = 1
c[1] = 1
for k ← 2 to n
        c[k] ← 0
        for i ← 1 to k
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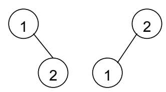
return c[n]
```



c[0]*c[1] + c[1]*c[0]

1 1

0 1 2 3 4 5 … n

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1 1 2

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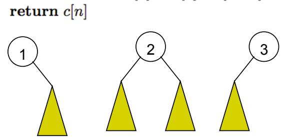
```



c[0]*c[2] + c[1]*c[1] + c[2]*c[0]

1 1 2

0 1 2 3 4 5 … n

```
BST-COUNT-DP(n)
c[0] = 1
c[1] = 1
for k ← 2 to n
        c[k] ← 0
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```

1 1 2 5

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return c[n]
```

1	1	2	5	…			
0	1	2	3	4	5	…	n

Running time?

```
BST-Count-DP(n)
c[0] = 1
c[1] = 1
for k ← 2 to n
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        for i ← 1 to k
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return c[n]
```

$\Theta(n^2)$

Longest common subsequence (LCS)

For a sequence $X = x_1, x_2, \ldots, x_n$, a subsequence is a subset of the sequence defined by a set of increasing indices $(i_1, i_2, \ldots, i_k)$ where $1 \le i_1 < i_2 < \ldots < i_k \le n$

X = A B A C D A B A B

ABA?

Longest common subsequence (LCS)

For a sequence $X = x_1, x_2, \ldots, x_n$, a subsequence is a subset of the sequence defined by a set of increasing indices $(i_1, i_2, \ldots, i_k)$ where $1 \le i_1 < i_2 < \ldots < i_k \le n$

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DCA?

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~~DCA~~

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X = A B A C D A B A B

AADAA?

Longest common subsequence (LCS)

For a sequence $X = x_1, x_2, \ldots, x_n$, a subsequence is a subset of the sequence defined by a set of increasing indices $(i_1, i_2, \ldots, i_k)$ where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$

X = A B A C D A B A B

AADAA

LCS problem

Given two sequences X and Y, a **common subsequence** is a subsequence that occurs in both X and Y

Given two sequences $X = x_1, x_2, \ldots, x_n$ and $Y = y_1, y_2, \ldots, y_n$,

What is the longest common subsequence?

X = A B C B D A B

Y = B D C A B A

LCS problem

Given two sequences X and Y, a **common subsequence** is a subsequence that occurs in both X and Y

Given two sequences $X = x_1, x_2, \ldots, x_n$ and $Y = y_1, y_2, \ldots, y_n$,

What is the longest common subsequence?

X = A B C B D A B

Y = B D C A B A

Step 1: Define the problem with respect to subproblems

X = A B C B D A B

Y = B D C A B A

Assume you have a solver for smaller problems

Step 1: Define the problem with respect to subproblems

X = A B C B D A ?

Y = B D C A B ?

Is the last character part of the LCS?

Step 1: Define the problem with respect to subproblems

X = A B C B D A ?

Y = B D C A B ?

Two cases: either the characters are the same or they're different

Step 1: Define the problem with respect to subproblems

X = $\boxed{\text{A B C B D A}}$ A

LCS

Y = $\boxed{\text{B D C A B}}$ A

The characters are part of the LCS

What is the recursive relationship?

If they're the same

$$LCS(X,Y) = LCS(X_{1...n-1}, Y_{1...m-1}) + x_n$$

Step 1: Define the problem with respect to subproblems

X = A B C B D A B

LCS

Y = B D C A B A

If they're different

$$LCS(X,Y) = LCS(X_{1...n-1},Y)$$

Step 1: Define the problem with respect to subproblems

X = A B C B D A B

LCS

Y = B D C A B A

If they're different

$$LCS(X,Y) = LCS(X,Y_{1...m-1})$$

Step 1: Define the problem with respect to subproblems

X = A B C B D A B

Y = B D C A B A

X = A B C B D A B

Y = B D C A B A

?

If they're different

Step 1: Define the problem with respect to subproblems

X = A B C B D A B

Y = B D C A B A

$$LCS(X,Y) = \begin{cases} 1 + LCS(X_{1...n-1},Y_{1...m-1}) & \text{if } x_n = y_m \\ \max(LCS(X_{1...n-1},Y), LCS(X,Y_{1...m-1})) & \text{otherwise} \end{cases}$$

(for now, let's just worry about counting the length of the LCS)

Step 2: Build the solution from the bottom up

$$LCS(X,Y) = \begin{cases} 1 + LCS(X_{1...n-1}, Y_{1...m-1}) & \text{if } x_n = y_m \\ \max(LCS(X_{1...n-1}, Y), LCS(X, Y_{1...m-1})) & \text{otherwise} \end{cases}$$

What types of subproblem solutions do we need to store?

LCS($X_{1...j}$, $Y_{1...k}$)

two different indices

Step 2: Build the solution from the bottom up

$$LCS(X,Y) = \begin{cases} 1 + LCS(X_{1...n-1}, Y_{1...m-1}) & \text{if } x_n = y_m \\ \max(LCS(X_{1...n-1}, Y), LCS(X, Y_{1...m-1})) & \text{otherwise} \end{cases}$$

What types of subproblem solutions do we need to store?

LCS($X_{1...j}$, $Y_{1...k}$)

$$LCS[i,j] = \begin{cases} 1 + LCS[i-1, j-1] & \text{if } x_i = y_j \\ \max(LCS[i-1, j], LCS[i, j-1]) & \text{otherwise} \end{cases}$$

$$LCS[i,j] = \begin{cases} 1 + LCS[i-1, j-1] & \text{if } x_i = y_j \\ \max(LCS[i-1, j], LCS[i, j-1]) & \text{otherwise} \end{cases}$$

i	j	0	1	2	3	4	5	6
	y_j		B	D	C	A	B	A
0	x_i							
1	A							
2	B							
3	C							
4	B							
5	D							
6	A							
7	B							

For Fibonacci and tree counting, we had to initialize some entries in the array. Any here?

$$LCS[i,j] = \begin{cases} 1 + LCS[i-1, j-1] & \text{if } x_i = y_j \\ \max(LCS[i-1, j], LCS[i, j-1]) & \text{otherwise} \end{cases}$$

i	j	0	1	2	3	4	5	6
	y_j		B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0						
2	B	0						
3	C	0						
4	B	0						
5	D	0						
6	A	0						
7	B	0						

Need to initialize values within 1 smaller in either dimension.

$$LCS[i,j]=\begin{cases}1+LCS[i-1,j-1] & \text{if } x_i=y_j\\ \boxed{\max(LCS[i-1,j],LCS[i,j-1]) \quad \text{otherwise}}\end{cases}$$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	?					
2	B	0						
3	C	0						
4	B	0						
5	D	0						
6	A	0						
7	B	0						

LCS(A, B)

$$LCS[i,j]=\begin{cases}1+LCS[i-1,j-1] & \text{if } x_i=y_j\\ \max(LCS[i-1,j],LCS[i,j-1]) & \text{otherwise}\end{cases}$$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0					
2	B	0						
3	C	0						
4	B	0						
5	D	0						
6	A	0						
7	B	0						

$$LCS[i,j]=\begin{cases}\boxed{1+LCS[i-1,j-1] \quad \text{if } x_i=y_j}\\ \max(LCS[i-1,j],LCS[i,j-1]) \quad \text{otherwise}\end{cases}$$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	?		
2	B	0						
3	C	0						
4	B	0						
5	D	0						
6	A	0						
7	B	0						

LCS(A, BDCA)

$$LCS[i,j]=\begin{cases}1+LCS[i-1,j-1] & \text{if } x_i=y_j\\ \max(LCS[i-1,j],LCS[i,j-1]) & \text{otherwise}\end{cases}$$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1		
2	B	0						
3	C	0						
4	B	0						
5	D	0						
6	A	0						
7	B	0						

LCS(A, BDCA)

$$LCS[i,j] = \begin{cases} 1 + LCS[i-1, j-1] & \text{if } x_i = y_j \\ \max(LCS[i-1, j], LCS[i, j-1]) & \text{otherwise} \end{cases}$$

i	j	0	1	2	3	4	5	6
	y_j		B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	?	
5	D	0						
6	A	0						
7	B	0						

LCS(ABCB, BDCAB)

$$LCS[i,j] = \begin{cases} 1 + LCS[i-1, j-1] & \text{if } x_i = y_j \\ \max(LCS[i-1, j], LCS[i, j-1]) & \text{otherwise} \end{cases}$$

i	j	0	1	2	3	4	5	6
	y_j		B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	
5	D	0						
6	A	0						
7	B	0						

LCS(ABCB, BDCAB)

$$LCS[i,j] = \begin{cases} 1 + LCS[i-1, j-1] & \text{if } x_i = y_j \\ \max(LCS[i-1, j], LCS[i, j-1]) & \text{otherwise} \end{cases}$$

i	j	0	1	2	3	4	5	6
	y_j		B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	3
6	A	0	1	2	2	3	3	4
7	B	0	1	2	2	3	4	4

Where's the final answer?

The algorithm

```
LCS-LENGTH(X, Y)
m ← length[X]
n ← length[Y]
c[0, 0] ← 0
for i ← 1 to m
        c[i, 0] ← 0
for j ← 1 to n
        c[0, j] ← 0
for i ← 1 to m
        for j ← 1 to n
                if x_i = y_i
                        c[i, j] ← 1 + c[i − 1, j − 1]
                elseif c[i − 1, j] > c[i, j − 1]
                        c[i, j] ← c[i − 1, j]
                else
                        c[i, j] ← c[i, j − 1]
return c[m, n]
```

The algorithm

```
LCS-LENGTH(X, Y)
m ← length[X]
n ← length[Y]
c[0,0] ← 0
for i ← 1 to m
        c[i,0] ← 0
for j ← 1 to n
        c[0,j] ← 0
for i ← 1 to m
        for j ← 1 to n
                if x_i = y_i
                        c[i,j] ← 1 + c[i − 1, j − 1]
                elseif c[i − 1, j] > c[i, j − 1]
                        c[i,j] ← c[i − 1, j]
                else
                        c[i,j] ← c[i, j − 1]
return c[m,n]
```

Base case initialization

The algorithm

```
LCS-LENGTH(X, Y)
m ← length[X]
n ← length[Y]
c[0,0] ← 0
for i ← 1 to m
        c[i,0] ← 0
for j ← 1 to n
        c[0,j] ← 0
for i ← 1 to m
        for j ← 1 to n
                if x_i = y_i
                        c[i,j] ← 1 + c[i − 1, j − 1]
                elseif c[i − 1, j] > c[i, j − 1]
                        c[i,j] ← c[i − 1, j]
                else
                        c[i,j] ← c[i, j − 1]
return c[m,n]
```

Fill in the matrix

The algorithm

```
LCS-LENGTH(X, Y)
m ← length[X]
n ← length[Y]
c[0,0] ← 0
for i ← 1 to m
        c[i,0] ← 0
for j ← 1 to n
        c[0,j] ← 0
for i ← 1 to m
        for j ← 1 to n
                if x_i = y_i
                        c[i,j] ← 1 + c[i − 1, j − 1]
                elseif c[i − 1, j] > c[i, j − 1]
                        c[i,j] ← c[i − 1, j]
                else
                        c[i,j] ← c[i, j − 1]
return c[m,n]
```

The algorithm

```
LCS-LENGTH(X, Y)
m ← length[X]
n ← length[Y]
c[0,0] ← 0
for i ← 1 to m
        c[i,0] ← 0
for j ← 1 to n
        c[0,j] ← 0
for i ← 1 to m
        for j ← 1 to n
                if x_i = y_i
                        c[i,j] ← 1 + c[i − 1, j − 1]
                elseif c[i − 1, j] > c[i, j − 1]
                        c[i,j] ← c[i − 1, j]
                else
                        c[i,j] ← c[i, j − 1]
return c[m,n]
```

The algorithm

```
LCS-LENGTH(X, Y)
m ← length[X]
n ← length[Y]
c[0, 0] ← 0
for i ← 1 to m
        c[i, 0] ← 0
for j ← 1 to n
        c[0, j] ← 0
for i ← 1 to m
        for j ← 1 to n
                if x_i = y_i
                        c[i, j] ← 1 + c[i − 1, j − 1]
                elseif c[i − 1, j] > c[i, j − 1]
                        c[i, j] ← c[i − 1, j]
                else
                        c[i, j] ← c[i, j − 1]
return c[m, n]
```

Running time?

```
LCS-LENGTH(X, Y)
m ← length[X]
n ← length[Y]
c[0, 0] ← 0
for i ← 1 to m
        c[i, 0] ← 0
for j ← 1 to n
        c[0, j] ← 0
for i ← 1 to m
        for j ← 1 to n
                if x_i = y_i
                        c[i, j] ← 1 + c[i − 1, j − 1]
                elseif c[i − 1, j] > c[i, j − 1]
                        c[i, j] ← c[i − 1, j]
                else
                        c[i, j] ← c[i, j − 1]
return c[m, n]
```

$\Theta(nm)$

Keeping track of the solution

Our LCS algorithm only calculated the length of the LCS between X and Y

What if we wanted to know the actual sequence?

Keep track of this as well…

```
for i ← 1 to m
        for j ← 1 to n
                if x_i = y_i
                        c[i, j] ← 1 + c[i − 1, j − 1]      ↖
                elseif c[i − 1, j] > c[i, j − 1]
                        c[i, j] ← c[i − 1, j]              ↑
                else
                        c[i, j] ← c[i, j − 1]              ←
return c[m, n]
```

$$LCS[i,j] = \begin{cases} 1 + LCS[i,j] & \text{if } x_i = y_j \\ \max(LCS[i-1,j], LCS[i,j-1]) & \text{otherwise} \end{cases}$$

i	j	0	1	2	3	4	5	6
	y_j		B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	3
6	A	0	1	2	2	3	3	4
7	B	0	1	2	2	3	4	4

We can follow the arrows to generate the solution

$$LCS[i,j] = \begin{cases} 1 + LCS[i,j] & \text{if } x_i = y_j \\ \max(LCS[i-1,j], LCS[i,j-1] & \text{otherwise} \end{cases}$$

i		j: 0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	3
6	A	0	1	2	2	3	3	4
7	B	0	1	2	2	3	4	4

We can follow the arrows to generate the solution

BCBA