BACKPROPAGATION

David Kauchak
CS158 – Fall 2019

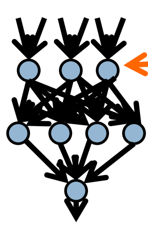
Admin

Grading!

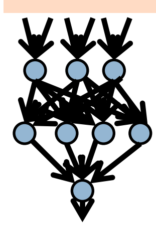
Assignment 7

Assignment 8

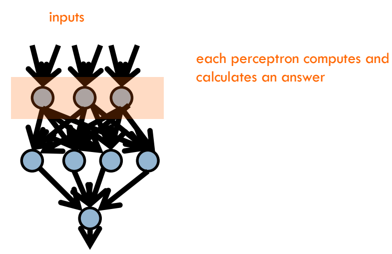
Neural network



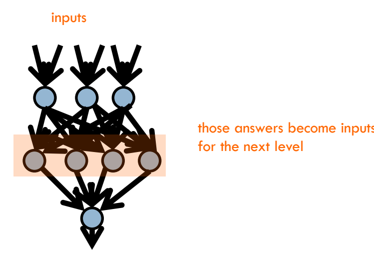
Neural network



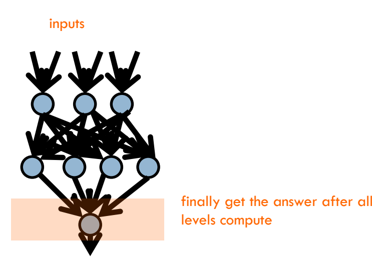
Neural network



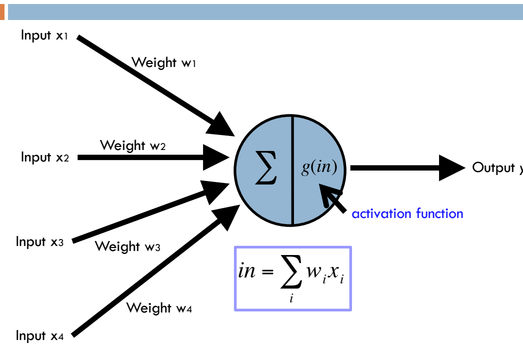
Neural network



Neural network



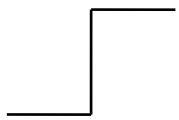
A neuron/perceptron



Activation functions

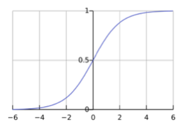
hard threshold:

$$g(in) = \begin{cases} 1 & \text{if } in > -b \\ 0 & otherwise \end{cases}$$

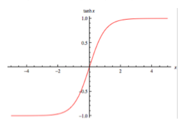


sigmoid

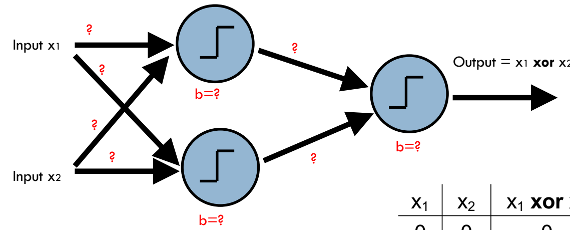
$$g(x) = \frac{1}{1+e^{-x}}$$



tanh x



Training

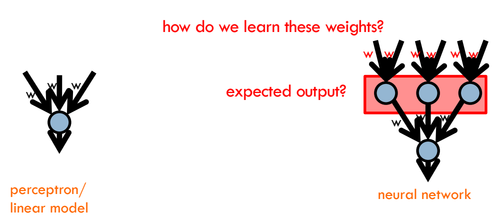


x_1	x_2	x_1 **xor** x_2
0	0	0
0	1	1
1	0	1
1	1	0

How do we learn the weights?

Learning in multilayer networks

Challenge: for multilayer networks, we don't know what the expected output/error is for the internal nodes!

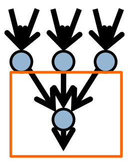


Backpropagation: intuition

Gradient descent method for learning weights by optimizing a loss function

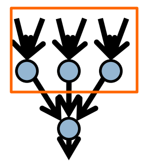
1. calculate output of all nodes
2. calculate the weights for the output layer based on the error
3. "backpropagate" errors through hidden layers

Backpropagation: intuition



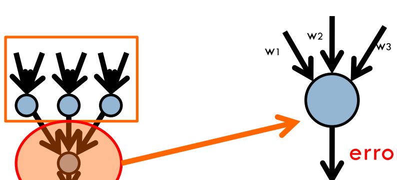
We can calculate the actual error here

Backpropagation: intuition



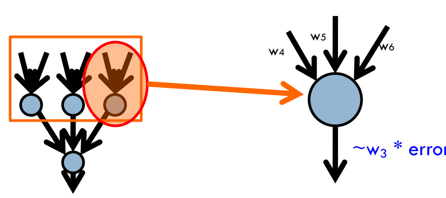
Key idea: propagate the error back to this layer

Backpropagation: intuition



error for node is ~ w_i * **error**

Backpropagation: intuition



Calculate as normal, but weight the error

Backpropagation: the details

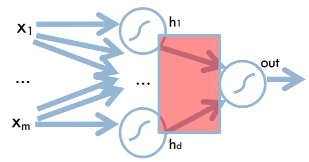
Gradient descent method for learning weights by optimizing a loss function

1. calculate output of all nodes
2. calculate the updates directly for the output layer
3. "backpropagate" errors through hidden layers

$$loss = \sum_x \frac{1}{2}(y - \hat{y})^2$$ squared error

Backpropagation: the details

Notation:



m: features/inputs

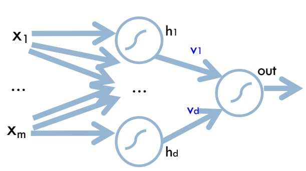
d: hidden nodes

h_k: output from hidden nodes

How many weights (ignore bias for now)?

Backpropagation: the details

Notation:



m: features/inputs

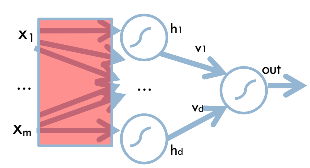
d: hidden nodes

h_k: output from hidden nodes

d weights: denote v_k

Backpropagation: the details

Notation:



m: features/inputs

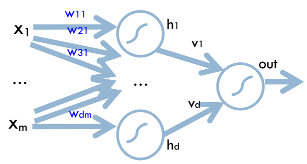
d: hidden nodes

h_k: output from hidden nodes

How many weights?

Backpropagation: the details

Notation:



m: features/inputs

d: hidden nodes

h_k: output from hidden nodes

d * m: denote w_{kj}

first index = hidden node
second index = feature

- w_{23}: weight from input 3 to hidden node 2
- w_4: all the *m* weights associated with hidden node 4

Backpropagation: the details

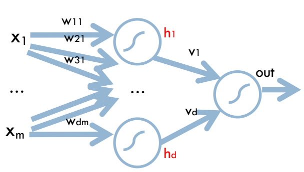
Gradient descent method for learning weights by optimizing a loss function

$$\text{argmin}_{w,v} \sum_x \frac{1}{2}(y - \hat{y})^2$$

1. calculate output of all nodes
2. calculate the updates directly for the output layer
3. "backpropagate" errors through hidden layers

Backpropagation: the details

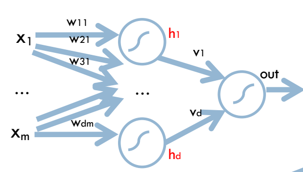
1. Calculate outputs of all nodes



What are h_k in terms of x and w?

Backpropagation: the details

1. Calculate outputs of all nodes



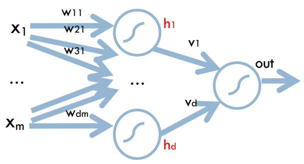
$$w_k \cdot x = \sum_{j=1}^{m} w_{kj} x_j$$

$$h_k = f(w_k \cdot x)$$

f is the activation function

Backpropagation: the details

1. Calculate outputs of all nodes

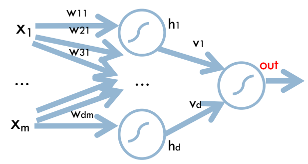


$$h_k = f(w_k \cdot x) = \frac{1}{1+e^{-w_k \cdot x}}$$

f is the activation function

Backpropagation: the details

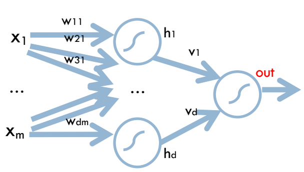
1. Calculate outputs of all nodes



What is *out* in terms of *h* and *v*?

Backpropagation: the details

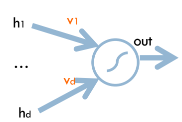
1. Calculate outputs of all nodes



$$out = f(v \cdot h) = \frac{1}{1+e^{-v \cdot h}}$$

Backpropagation: the details

2. Calculate new weights for output layer

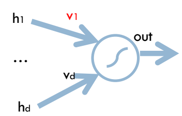


$$\text{argmin}_{w,v} \sum_x \frac{1}{2}(y-\hat{y})^2$$

Want to take a small step towards decreasing loss

Output layer weights

$$\text{argmin}_{w,v} \sum_x \frac{1}{2}(y - \hat{y})^2$$



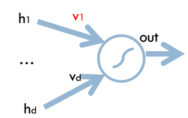
$$\frac{dloss}{dv_k} = \frac{d}{dv_k}\left(\frac{1}{2}(y - \hat{y})^2\right)$$

$$= \frac{d}{dv_k}\left(\frac{1}{2}(y - f(v \cdot h))^2\right) \qquad \hat{y} = f(v \cdot h)$$

$$= (y - f(v \cdot h))\frac{d}{dv_k}(y - f(v \cdot h))$$

Output layer weights

$$= (y - f(v \cdot h))\frac{d}{dv_k}(y - f(v \cdot h))$$



$$= -(y - f(v \cdot h))\frac{d}{dv_k} f(v \cdot h)$$

$$= -(y - f(v \cdot h)) f'(v \cdot h)\frac{d}{dv_k} v \cdot h$$

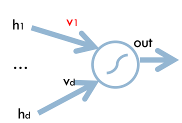
$$= -(y - f(v \cdot h)) f'(v \cdot h) h_k \qquad v \cdot h = \sum_k v_k h_k$$

The actual update is a step towards *decreasing* loss:

$$v_k = v_k + (y - f(v \cdot h)) f'(v \cdot h) h_k$$

Output layer weights

$$v_k = v_k + \underbrace{(y - f(v \cdot h))}\underbrace{f'(v \cdot h)}\underbrace{h_k}$$

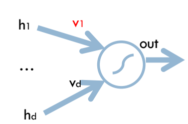


What are each of these?

Do they make sense individually?

Output layer weights

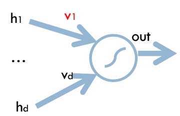
$$v_k = v_k + \underbrace{(y - f(v \cdot h))}\underbrace{f'(v \cdot h)}\underbrace{h_k}$$



- $(y - f(v \cdot h))$: how far from correct and which direction
- $f'(v \cdot h)$: slope of the activation function where input is at
- h_k: size and direction of the feature associated with this weight

Output layer weights

$$v_k = v_k + (y - f(v \cdot h)) f'(v \cdot h) h_k$$



how far from correct
and which direction

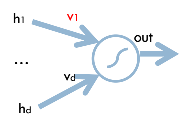
$$(y - f(v \cdot h)) > 0$$

?

$$(y - f(v \cdot h)) < 0$$

Output layer weights

$$v_k = v_k + (y - f(v \cdot h)) f'(v \cdot h) h_k$$



how far from correct
and which direction

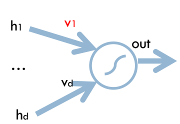
$(y - f(v \cdot h)) > 0$ prediction < label: increase the weight

$(y - f(v \cdot h)) < 0$ prediction > label: decrease the weight

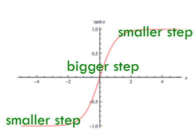
bigger difference = bigger change

Output layer weights

$$v_k = v_k + (y - f(v \cdot h)) f'(v \cdot h) h_k$$

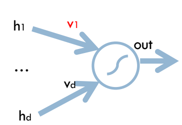


slope of the activation
function where input is at



Output layer weights

$$v_k = v_k + (y - f(v \cdot h)) f'(v \cdot h) h_k$$



size and direction of the
feature associated with
this weight

perceptron update:

$$w_j = w_j + x_{ij} y_i$$

gradient descent update:

$$w_j = w_j + x_{ij} y_i c$$

Backpropagation: the details

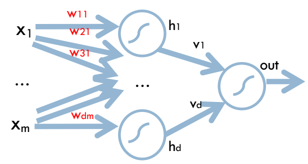
Gradient descent method for learning weights by optimizing a loss function

$$\text{argmin}_{w,v} \sum_x \frac{1}{2}(y-\hat{y})^2$$

1. calculate output of all nodes
2. calculate the updates directly for the output layer
3. "backpropagate" errors through hidden layers

Backpropagation

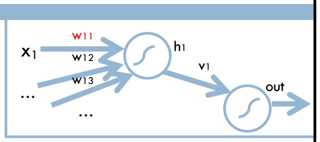
3. "backpropagate" errors through hidden layers



$$\text{argmin}_{w,v} \sum_x \frac{1}{2}(y-\hat{y})^2$$

Want to take a small step towards decreasing loss

Hidden layer weights



$$\frac{dloss}{dw_{kj}} = \frac{d}{dw_{kj}}\left(\frac{1}{2}(y-\hat{y})^2\right)$$

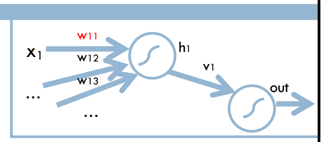
$$= \frac{d}{dw_{kj}}\left(\frac{1}{2}(y-f(v\cdot h))^2\right) \qquad \hat{y} = f(v\cdot h)$$

$$= (y-f(v\cdot h))\frac{d}{dw_{kj}}\left(y-f(v\cdot h)\right)$$

$$= -(y-f(v\cdot h))\frac{d}{dw_{kj}}f(v\cdot h)$$

$$= -(y-f(v\cdot h))f'(v\cdot h)\frac{d}{dw_{kj}}v\cdot h$$

Hidden layer weights



$$= -(y-f(v\cdot h))f'(v\cdot h)\frac{d}{dw_{kj}}v\cdot h$$

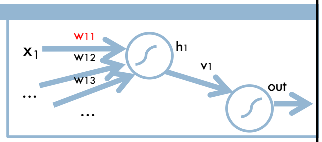
$$= -(y-f(v\cdot h))f'(v\cdot h)\frac{d}{dw_{kj}}v_k h_k$$

derivative of other vh components are not affected by w_{kj}

$$= -(y-f(v\cdot h))f'(v\cdot h)v_k\frac{d}{dw_{kj}}h_k$$

$$= -(y-f(v\cdot h))f'(v\cdot h)v_k\frac{d}{dw_{kj}}f(w_k\cdot x)$$

Hidden layer weights



$$= -(y - f(v \cdot h)) f'(v \cdot h) v_k \frac{d}{dw_{kj}} f(w_k \cdot x)$$

$$= -(y - f(v \cdot h)) f'(v \cdot h) v_k f'(w_k \cdot x) \frac{d}{dw_{kj}} w_k \cdot x$$

$$= -(y - f(v \cdot h)) f'(v \cdot h) v_k f'(w_k \cdot x) x_j \qquad w_k \cdot x = \sum_j w_{kj} x_j$$

Why all the math?

"More math! High level stuff is interesting, but having a strong grasp of the underlying math lets you reason out what's going on."

"It feels like it is really shying away from the mathematical background at times in the interest of not overwhelming us, but it would be best (in my opinion) if we got the math we need to fully justify the approach."

$$\frac{dloss}{dv_k} = \frac{d}{dv_k}\left(\frac{1}{2}(y - \hat{y})^2\right)$$

$$= \frac{d}{dv_k}\left(\frac{1}{2}(y - f(v \cdot h)^2\right)$$

$$= (y - f(v \cdot h)) \frac{d}{dv_k}\left(y - f(v \cdot h)\right)$$

$$= -(y - f(v \cdot h)) \frac{d}{dv_k} f(v \cdot h)$$

$$= -(y - f(v \cdot h)) f'(v \cdot h) \frac{d}{dv_k} v \cdot h$$

What happened here?

$$= -(y - f(v \cdot h)) f'(v \cdot h) h_k$$

$$\frac{dloss}{dw_{kj}} = \frac{d}{dw_{kj}}\left(\frac{1}{2}(y - \hat{y})^2\right)$$

$$= \frac{d}{dw_{kj}}\left(\frac{1}{2}\left(y - f(v \cdot h)^2\right)\right)$$

$$= (y - f(v \cdot h)) \frac{d}{dw_{kj}}\left(y - f(v \cdot h)\right)$$

$$= -(y - f(v \cdot h)) \frac{d}{dw_{kj}} f(v \cdot h)$$

$$= -(y - f(v \cdot h)) f'(v \cdot h) \frac{d}{dw_{kj}} v \cdot h$$

$$= -(y - f(v \cdot h)) f'(v \cdot h) \frac{d}{dw_{kj}} v_k h_k$$

$$= -(y - f(v \cdot h)) f'(v \cdot h) v_k \frac{d}{dw_{kj}} h_k$$

$$= -(y - f(v \cdot h)) f'(v \cdot h) v_k \frac{d}{dw_{kj}} f(w_k \cdot x)$$

$$= -(y - f(v \cdot h)) f'(v \cdot h) v_k f'(w_k \cdot x) \frac{d}{dw_{kj}} w_k \cdot x$$

$$= -(y - f(v \cdot h)) f'(v \cdot h) v_k f'(w_k \cdot x) x_j$$

$$= -(y - f(v \cdot h)) f'(v \cdot h) \frac{d}{dw_{kj}} v \cdot h$$

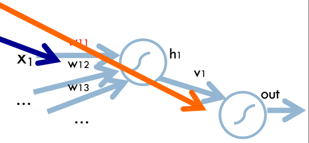
$$= -(y - f(v \cdot h)) f'(v \cdot h) \frac{d}{dw_{kj}} v_k h_k$$

$$= -(y - f(v \cdot h)) f'(v \cdot h) v_k \frac{d}{dw_{kj}} h_k$$

$$= -(y - f(v \cdot h)) f'(v \cdot h) v_k \frac{d}{dw_{kj}} f(w_k \cdot x)$$

$$= -(y - f(v \cdot h)) f'(v \cdot h) v_k f'(w_k \cdot x) \frac{d}{dw_{kj}} w_k \cdot x$$

$$= -(y - f(v \cdot h)) f'(v \cdot h) v_k f'(w_k \cdot x) x_j$$



What is the slope vh with respect to w_{kj}

Backpropagation

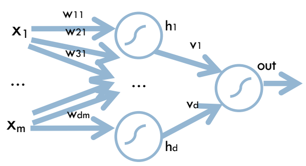
output layer

$$= -(y - f(v \cdot h)) f'(v \cdot h) h_k$$

hidden layer

$$= -(y - f(v \cdot h)) f'(v \cdot h) v_k f'(w_k \cdot x) x_j$$

What's different?



Backpropagation

output layer

$$= -(y - f(v \cdot h)) f'(v \cdot h) h_k$$

error

output activation slope

input

hidden layer

$$= -(y - f(v \cdot h)) f'(v \cdot h) v_k f'(w_k \cdot x) x_j$$

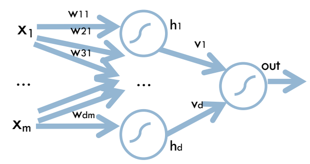
error

output activation slope

input

weight from hidden layer to output layer

slope of wx



Backpropagation

output layer

$$= -(y - f(v \cdot h)) f'(v \cdot h) h_k$$

error

output activation slope

input

hidden layer

$$= -(y - f(v \cdot h)) f'(v \cdot h) v_k f'(w_k \cdot x) x_j$$

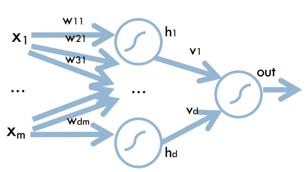
error

output activation slope

input

how much of the error came from this hidden node

how much do we need to change



Backpropgation generalization

output layer

$$v_k = v_k + (y - f(v \cdot h)) f'(v \cdot h) h_k$$

$$= v_k + h_k (y - f(v \cdot h)) f'(v \cdot h)$$

$$v_k = v_k + h_k \Delta_{out}$$

$$\Delta_{out} = f'(v \cdot h)(y - f(v \cdot h))$$

modified error

derivative of input at node

error

Backpropgation generalization

output layer

$$v_k = v_k + (y - f(v \cdot h)) f'(v \cdot h) h_k$$
$$= v_k + h_k (y - f(v \cdot h)) f'(v \cdot h)$$

$$v_k = v_k + h_k \Delta_{out}$$
$$\Delta_{out} = f'(v \cdot h)(y - f(v \cdot h))$$

hidden layer

$$w_{kj} = w_{kj} + (y - f(v \cdot h)) f'(v \cdot h) v_k f'(w_k \cdot x) x_j$$
$$= w_{kj} + x_j f'(w_k \cdot x) v_k f'(v \cdot h)(y - f(v \cdot h))$$

$$w_{kj} = w_{kj} + x_j \Delta_k$$
$$\Delta_k = f'(w_k \cdot x) v_k f'(v \cdot h)(y - f(v \cdot h))$$

Can we write this more succinctly?

Backpropgation generalization

output layer

$$v_k = v_k + (y - f(v \cdot h)) f'(v \cdot h) h_k$$
$$= v_k + h_k (y - f(v \cdot h)) f'(v \cdot h)$$

$$v_k = v_k + h_k \Delta_{out}$$
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$$\Delta_k = f'(w_k \cdot x) v_k f'(v \cdot h)(y - f(v \cdot h))$$
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Backpropgation generalization

output layer

$$v_k = v_k + h_k \Delta_{out}$$
$$\Delta_{out} = f'(v \cdot h)(y - f(v \cdot h))$$

hidden layer

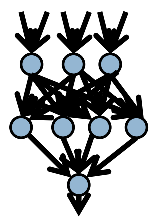
$$w_{kj} = w_{kj} + x_j \Delta_k$$
$$\Delta_k = f'(w_k \cdot x) v_k f'(v \cdot h)(y - f(v \cdot h))$$
$$= f'(w_k \cdot x) v_k \Delta_{out}$$

weight to output layer

modified error of output layer

$$w = w + input * \Delta_{current}$$
$$\Delta_{current} = f'(current\_input) w_{output} \Delta_{output}$$

Backprop on multilayer networks

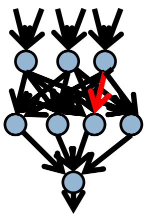


Anything different here?

$$w = w + input * \Delta_{current}$$
$$\Delta_{current} = f'(current\_input) w_{output} \Delta_{output}$$

$$w = w + input * \Delta_{output}$$

Backprop on multilayer networks

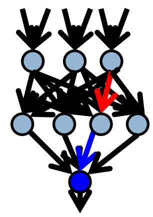


$$w = w + input * \Delta_{current}$$
$$\Delta_{current} = f'(current\_input) w_{output} \Delta_{output}$$

$$w = w + input * \Delta_{output}$$

What "errors" at the next layer does the highlighted edge affect?

Backprop on multilayer networks



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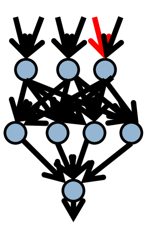
Backprop on multilayer networks



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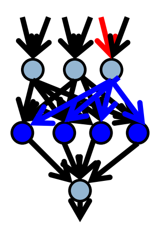
Backprop on multilayer networks



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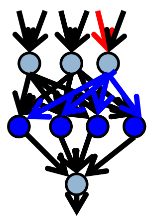
Backprop on multilayer networks



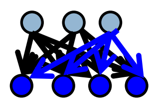
$w = w + input * \Delta_{current}$
$\Delta_{current} = f'(current\_input)\sum w_{output}\Delta_{output}$

$w = w + input * \Delta_{current}$
$\Delta_{current} = f'(current\_input) w_{output}\Delta_{output}$

$w = w + input * \Delta_{output}$

Backprop on multilayer networks

...



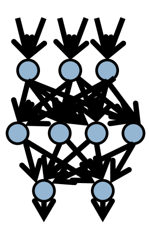
...

$w = w + input * \Delta_{current}$
$\Delta_{current} = f'(current\_input)\sum w_{output}\Delta_{output}$

Backpropogation:

- Calculate new weights and modified errors at output layer
- Recursively calculate new weights and modified errors on hidden layers based on recursive relationship
- Update model with new weights

Multiple output nodes



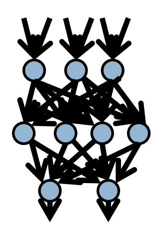
$w = w + input * \Delta_{current}$
$\Delta_{current} = f'(current\_input)\sum w_{output}\Delta_{output}$

$w = w + input * \Delta_{current}$
$\Delta_{current} = f'(current\_input) w_{output}\Delta_{output}$

$w = w + input * \Delta_{output}$

How does multiple outputs change things?

Multiple output nodes



$w = w + input * \Delta_{current}$
$\Delta_{current} = f'(current\_input)\sum w_{output}\Delta_{output}$

$w = w + input * \Delta_{current}$
$\Delta_{current} = f'(current\_input)\sum w_{output}\Delta_{output}$

$w = w + input * \Delta_{output}$

How does multiple outputs change things?

Backpropagation implementation

Output layer update:

$$v_k = v_k + h_k(y - f(v \cdot h))f'(v \cdot h)$$

Hidden layer update:

$$w_{kj} = w_{kj} + x_j f'(w_k \cdot x) v_k f'(v \cdot h)(y - f(v \cdot h))$$

Any missing information for implementation?

Backpropagation implementation

Output layer update:

$$v_k = v_k + h_k(y - f(v \cdot h))f'(v \cdot h)$$

Hidden layer update:

$$w_{kj} = w_{kj} + x_j f'(w_k \cdot x) v_k f'(v \cdot h)(y - f(v \cdot h))$$

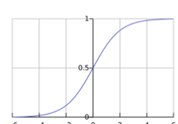
1. What activation function are we using
2. What is the derivative of that activation function

Activation function derivatives

sigmoid

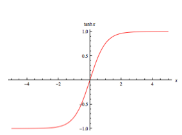
$$s(x) = \frac{1}{1 + e^{-x}}$$

$$s'(x) = s(x)(1 - s(x))$$



tanh

$$\frac{d}{dx}\tanh(x) = 1 - \tanh^2 x$$



Learning rate

Output layer update:

$$v_k = v_k + \eta h_k(y - f(v \cdot h))f'(v \cdot h)$$

Hidden layer update:

$$w_{kj} = w_{kj} + \eta x_j f'(w_k \cdot x) v_k f'(v \cdot h)(y - f(v \cdot h))$$

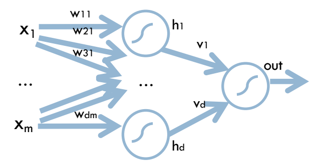
- Like gradient descent for linear classifiers, use a learning rate
- Often will start larger and then get smaller

Backpropagation implementation

Just like gradient descent!

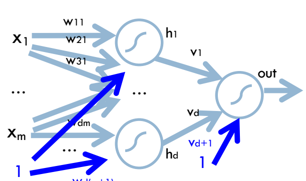
for some number of iterations:
 randomly shuffle training data
 for each example:
 - Compute all outputs going forward
 - Calculate new weights and modified errors at output layer
 - Recursively calculate new weights and modified errors on hidden layers based on recursive relationship
 - Update model with new weights

Handling bias



How should we learn the bias?

Handling bias



1. Add an extra feature hard-wired to 1 to all the examples
2. For other layers, add an extra parameter whose input is always 1

Online vs. batch learning

for some number of iterations:
 randomly shuffle training data
 for each example:
 - Compute all outputs going forward
 - Calculate new weights and modified errors at output layer
 - Recursively calculate new weights and modified errors on hidden layers based on recursive relationship
 - Update model with new weights

Online learning: update weights after each example

Batch learning?

Batch learning

for some number of iterations:

randomly shuffle training data

initialize weight accumulators to 0 (one for each weight)

for each example:

- Compute all outputs going forward
- Calculate new weights and modified errors at output layer
- Recursively calculate new weights and modified errors on hidden layers based on recursive relationship
- Add new weights to weight accumulators

Divide weight accumulators by number of examples

Update model weights by weight accumulators

Process *all* of the examples before updating the weights

Many variations

Momentum: include a factor in the weight update to keep moving in the direction of the previous update

Mini-batch:

- Compromise between online and batch
- Avoids noisiness of updates from online while making more educated weight updates

Simulated annealing:

- With some probability make a random weight update
- Reduce this probability over time

…

Challenges of neural networks?

Picking network configuration

Can be slow to train for large networks and large amounts of data

Loss functions (including squared error) are generally not convex *with respect to the parameter space*

History of Neural Networks

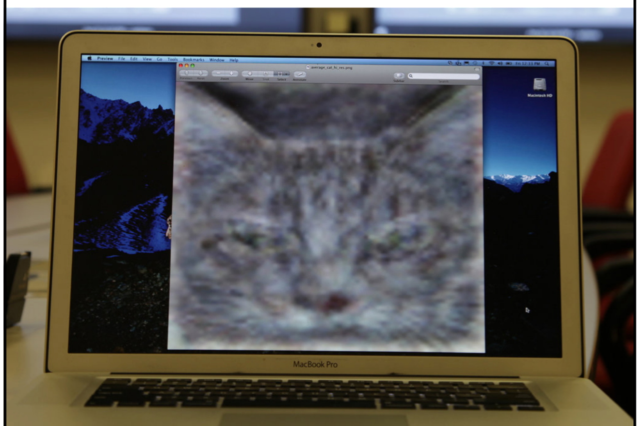
McCulloch and Pitts (1943) – introduced model of artificial neurons and suggested they could learn

Hebb (1949) – Simple updating rule for learning

Rosenblatt (1962) - the *perceptron* model

Minsky and Papert (1969) – wrote *Perceptrons*

Bryson and Ho (1969, but largely ignored until 1980s--Rosenblatt) – invented backpropagation learning for multilayer networks



http://www.nytimes.com/2012/06/26/technology/in-a-big-network-of-computers-evidence-of-machine-learning.html?_r=0